ARL 72-0053
APRIL 1972

Aerospace Research Laboratories

REPRESENTATIONS OF THE LORENTZ GROUP: RECENT DEVELOPMENTS

MOSHE CARMELI
GENERAL PHYSICS RESEARCH LABORATORY
SHIMON MALIN
COLGATE UNIVERSITY
HAMILTON, NEW YORK

PROJECT NO. 7114

Approved for public release; distribution unlimited.

AIR FORCE SYSTEMS COMMAND
United States Air Force

NOTICES

AIR FORCE: 23-5-72/200

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) General Physics Research Laboratory Aerospace Research Laboratories Wright-Patterson AFB, Ohio 45433	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP LG-1

3. REPORT TITLE

"Representations of the Lorentz Group: Recent Developments"

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Internal Report

5. AUTHOR(S) (First name, middle initial, last name)

Moshe Carmeli
Shimon Malin

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
APRIL 1972	71 71	66

8a. ~~XXXXXXXXXXXXXX~~ In-House Research b. PROJECT NO. 7114-00-08 c. DOD Element: 61102F d. DOD Subelement: 681301	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) ARL 72-0053

10. DISTRIBUTION STATEMENT

Approved for public release - distribution unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
TECH OTHER	ARL/LG WPAFB, Ohio

13. ABSTRACT

Recent developments in the theory of representations of the Lorentz group, in which all infinite-dimensional representations of the group were expressed in spinor-like forms, are reviewed.

DD FORM 1 NOV 65 1473

UNCLASSIFIED
Security Classification

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT

☆U.S.Government Printing Office: 1972 — 759-085/567

Security Classification

ARL 72-0053

REPRESENTATIONS OF THE LORENTZ GROUP: RECENT DEVELOPMENTS

MOSHE CARMELI
GENERAL PHYSICS RESEARCH LABORATORY
AEROSPACE RESEARCH LABORATORIES

AND

*SHIMON MALIN**
DEPARTMENT OF PHYSICS AND ASTRONOMY
COLGATE UNIVERSITY, HAMILTON, NEW YORK 13346

**Supported in part by the Colgate Research Council and the Sloan Foundations.*

APRIL 1972

PROJECT 7114

Approved for public release; distribution unlimited.

AEROSPACE RESEARCH LABORATORIES
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433

FOREWORD

This Technical Report presents results of research carried out by Professor S. Malin, Colgate University, and Dr. M. Carmeli, General Physics Research Laboratory, Aerospace Research Laboratories. Dr. Carmeli's work was accomplished on Project No. 7114.

ABSTRACT

Recent developments in the theory of representations of the Lorentz group, in which all infinite-dimensional representations of the group were expressed in spinor-like forms, are reviewed.

TABLE OF CONTENTS

1. INTRODUCTION

Infinite-dimensional representations of noncompact Lie groups are currently of interest and are being studied in describing the physics of elementary particles.[1] Of particular importance in this class of groups is SL(2, C), the group of all 2x2 complex matrices with determinant unity. This is the covering group of the restricted Lorentz group describing homogeneous Lorentz transformations that are orthochroneous and proper.[2] This group plays an important role in relativistic quantum mechanics, quantum field theory, S-matrix theory and axiomatic field theory.

The theory of representations of the Lorentz group is of particular interest in connection with recent developments utilizing infinite-component wave equations to describe particle properties. This approach was originally attempted by Majorana (1932), who suggested an infinite-dimensional wave equation whose form is analogous to Dirac's spinor equation. It turns out, however, that Majorana's equation generates an unphysical mass specturm. The theory of infinite-component wave equations, which are generalizations of Dirac's equation, was subsequently developed by Gel'fand and Iaglom (1948).

Recently, Nambu, Barut, Fronsdal, and others[3] developed more complex types of equations based on the Lorentz group and showed how to describe particle properties within this framework. Barut and his co-workers, in particular Kleinert and Conigan, have also developed an approach

in which particle properties (mass spectrum, magnetic moments, form factors, etc.) are directly expressed in terms of operators in a given infinite-dimensional representation, without using wave equations.

Various aspects of the theory of infinite-component c-number wave-functions and wave equations were also investigated by Bohm (1967, 1968), Lam (1968), Miyazaki (1968a, b), Takabayashi (1967) and many others.[4] The relationship between Regge's theory and Lorentz invariance was investigated, and equations which reproduce Regge mass spectra were proposed.[5] The problem of second quantization of such theories, which raised some deep difficulties, were extensively investigated (Grodsky and Streater 1968; Abers, Grodsky, and Norton 1967; Feldmann and Matthews 1967a, b; Fronsdal 1967c; Oksak and Todorov 1969, 1970; Miyazaki 1970b, c). The relationship between infinite-dimensional wave-functions and the foundations of quantum mechanics was investigated by Barut and Malin (1968a, b, 1971). Infinite-dimensional representations were used in relation to the problem of representing the algebra of current density (Dashen and Gell-Mann 1966; Bebie and Lentwyler 1967; Lentwyler 1968; Gell-Mann, Horn and Weyers 1967; Barut and Komen 1970; Hamprect and Kleinert 1969; Kleinert, Corrigan, and Hamprect 1970; Cocho, Fronsdal, and White 1969; Fronsdal and Harun-Ar Rashid 1969; Chang, Dashen, and O'Raifeartaigh 1969a, b; Katz and Noga 1970). The applications of infinite-dimensional representations of the Lorentz group to particle physics were recently reviewed by Miyazaki (1970b).[6] All these problems are very complex both physically and mathematically.

As far as the mathematical theory of representations of the Lorentz group is concerned, there are essentially two approaches: (1) the infinitesimal approach, in which one finds the matrices corresponding to infinitesimal generators in a given representation and expresses matrices corresponding to finite group elements as exponential functions of the generators (Bargmann 1947); and (2) the global approach, in which the representations are realized as operators defined over an abstract space of functions (Gel'fand and Naimark 1946, 1947).

Recently, the authors (Carmeli 1970; Carmeli and Malin 1971, 1972) introduced a generalized Fourier transformation which enabled them to use the global approach for expressing infinite-dimensional representations in terms of matrices, generalizing the spinor[7] form of finite-dimensional representation to the infinite-dimensional case. While the usual spinor representations are non-unitary, this new form describes both unitary and non-unitary representations.

The purpose of the present review is to summarize these recent developments in the theory of representations of the Lorentz group.

Sections 2 and 3 include reviews of the infinitesimal approach and of the finite-dimensional representations of the group SL(2, C). The content of these sections is well known, but is given here for completeness and to establish the notation. The principal, complementary, and complete series of representations are then discussed in Sections 4, 5, and 6, respectively.

Throughout the paper we adopt the now standardized notation and terminology of Naimark (1964).

2. THE INFINITESIMAL APPROACH

A. Infinitesimal Lorentz Matrices

A linear transformation g of the variables x_1, x_2, x_3 and x_4 which leaves the form $x_1^2 + x_2^2 + x_3^2 - x_4^2$ invariant is called a *Lorentz transformation*. The aggregate of all such linear transformations g provides a group which is called the *Lorentz group*. If $g_{44} \geq 1$, the transformation is called *orthochroneous*. The aggregate of all orthochroneous Lorentz transformations provides a subgroup of the Lorentz group. The determinant of every Lorentz transformation is equal to either $+1$, in which case the transformation is called *proper*, or to -1, in which case it is called *improper*. The aggregate of all proper, orthochroneous Lorentz transformations also forms a group which is a subgroup of the Lorentz group.[8] *Throughout this paper we will be concerned with the group of all proper orthochroneous Lorentz transformations*. This group is denoted by L.

Rotations $a_1(\Psi)$, $a_2(\Psi)$, $a_3(\Psi)$ and Lorentz transformations $b_1(\Psi)$, $b_2(\Psi)$, $b_3(\Psi)$, around and along Ox_1, Ox_2, Ox_3 can then be written explicitly.[9] The infinitesimal matrices a_r and b_r of the group L are defined by[10]

$$a_r = \left[\frac{da_r(\Psi)}{d\Psi} \right]_{\Psi = 0}, \quad b_r = \left[\frac{db_r(\Psi)}{d\Psi} \right]_{\Psi = 0} \tag{2.1}$$

and satisfy the commutation relations

$$\begin{aligned} \left[a_i, a_j\right] &= \varepsilon_{ijk} a_k \\ \left[b_i, b_j\right] &= -\varepsilon_{ijk} a_k \\ \left[a_i, b_j\right] &= \varepsilon_{ijk} b_k . \end{aligned} \tag{2.2}$$

B. Infinitesimal Operators

We denote an arbitrary linear representation of the group L in a Banach space B by $g \to T_g$ and for convenience we put[11]

$$A_r(\Psi) = T_{a_r(\Psi)}, \quad B_r(\Psi) = T_{b_r(\Psi)}. \tag{2.3}$$

The basic infinitesimal operators of the one-parameter groups $A_r(\Psi)$ and $B_r(\Psi)$ are then defined by[12]

$$A_r = \left[\frac{dA_r(\Psi)}{d\Psi} \right]_{\Psi = 0}, \qquad B_r = \left[\frac{dB_r(\Psi)}{d\Psi} \right]_{\Psi = 0}, \tag{2.4}$$

if the representation is finite-dimensional. If the representation $g \to T_g$ is infinite-dimensional, however, the operator functions $A_r(\Psi)$ and $B_r(\Psi)$ might be non-differentiable, but there may still exist a vector x for which $A_r(\Psi)x$ and $B_r(\Psi)x$ are differentiable vector-functions.[13]

A representation $g \to T_g$ of the group L is completely determined by its infinitesimal operators A_i and B_i, $i = 1, 2, 3$. The determination of the irreducible representations of the group L is based on the fact that the basic infinitesimal operators of a representation satisfy the same commutation relations that exists among the infinitesimal matrices a_r and b_r:

$$\begin{aligned} \left[A_i, A_j\right] &= \epsilon_{ijk} A_k, \\ \left[B_i, B_j\right] &= -\epsilon_{ijk} A_k, \\ \left[A_i, B_j\right] &= \epsilon_{ijk} B_k. \end{aligned} \tag{2.5}$$

Defining now the operators

$$\begin{aligned} H_\mp &= iA \pm A_2, \quad H_3 = iA_3 \\ F_\mp &= iB_1 \pm B_2, \quad F_3 = iB_3, \end{aligned} \tag{2.6}$$

one finds

$$[H_\mp , H_3] = \pm H_\mp , \quad [H_+, H_-] = 2H_3 .$$

$$[F_\mp, F_3] = \mp H_\mp, \quad [F_+, F_-] = -2H_3$$

$$[H_\pm, F_\pm] = 0, \quad [H_3, F_3] = 0, \quad [H_\pm, F_\mp] = \pm 2F_3$$

$$[H_\pm, F_3] = \mp F_\pm, \quad [F_\pm, H_3] = \mp F_\pm . \qquad (2.7)$$

The problem then reduces to the determination of $H_\pm$, H_3, $F_\pm$, F_3 satisfying the conditions (2.7).

Now, since the three-dimensional pure rotation group O_3 is a subgroup of the proper, orthochroneous Lorentz group L, obviously every representation of L is also a representation of O_3. Clearly, if a given representation of L is irreducible it need not be irreducible when considered as a representation of O_3. In fact, any infinite-dimensional representation of L, when regarded as a representation of O_3, is highly reducible; it is equivalent to a direct sum of an infinite number of irreducible representations. The space R of any irreducible representation of the group L is, therefore, a closed direct sum of subspaces M^j, where M^j is the (2j+1)-dimensional space in which the irreducible representation of weight j of the group O_3 is realized.

Following the standard convention, one chooses the 2j+1 normalized eigenvectors of the operator H_3 as the canonical basis for the subspace M^j. Let these base vectors be denoted as f_m^j, where m = -j, -j+1, ..., j, the superscript j indicates the subspace to which f_m^j belongs,[15] and the subscript is the eigenvalue of the operator H_3. A detailed investigation of the commutation relations (2.7) in terms of the canonical basis f_m^j leads to the following conclusions:

(a) Each irreducible representation of the group L is characterized by a pair of numbers (j_0, c), where j_0 is integral or half-integral, and c is a complex number.

(b) The space $R(j_0, c)$ of any given irreducible infinite-dimensional representation of the group L is characterized by the integer or half-integer j_0 such that $R(j_0, c) = M^{j_0} \oplus M^{j_0+1} \oplus \dots$. The whole space $R(j_0, c)$ is spanned, therefore, by the set of base-vectors f_m^j, where $j = j_0, j_0+1, j_0+2, \dots$, and $m = -j, j+1, \dots, j$. If the given irreducible representation is finite-dimensional than the direct sum of the subspaces M's terminates after a finite number of terms.

(c) A given representation is finite-dimensional if and only if $c^2 = (j_0 + n)^2$, for some natural number n.

(d) The irreducible representation corresponding to a given pair (j_0, c) is, with a suitable choice of basis f_m^j in the space of representation, given by the formulas[15]

$$H_{\pm} f_m^j = [(j \pm m + 1)(j \mp m)]^{\frac{1}{2}} f_{m\pm 1}^j$$

$$H_3 f_m^j = m f_m^j$$

$$\begin{aligned} F_{\pm} f_m^j = {} & \pm [(j \mp m)(j \mp m-1)]^{\frac{1}{2}} C_j f_{m\pm 1}^{j-1} \\ & - [(j \mp m)(j \pm m + 1)]^{\frac{1}{2}} A_j f_{m\pm 1}^j \\ & \pm [(j \pm m + 1)(j \pm m + 2)]^{\frac{1}{2}} C_{j+1} f_{m\pm 1}^{j+1} \end{aligned}$$

$$\begin{aligned} F_3 f_m^j & [(j - m)(j + m)]^{\frac{1}{2}} C_j f_m^{j-1} - m A_j f_m^j \\ & - [(j + m + 1)(j - m + 1)]^{\frac{1}{2}} C_{j+1} f_m^{j+1} \end{aligned} \tag{2.8}$$

Here $A_j = i\, c\, j_0 / j(j+1)$, and $C_j = i\,(j^2 - j_0^2)^{\frac{1}{2}}\,(j^2 - c^2)^{\frac{1}{2}} / j\,(4j^2-1)^{\frac{1}{2}}$.

(e) To each pair of numbers (j_0, c), where j_0 is integral or half-integral and c is complex, there corresponds a representation $g \to T_g$ of the group L, whose infinitesimal operators are given by Eqs. (2.8).

C. Unitarity Conditions

If the representation $g \to T_g$ of the group L is unitary,[16,17] then Eqs. (2.8) satisfy certain conditions which are summarized below.

Let A be an infinitesimal operator of a unitary representation $g \to T_g$ of the group L. Then $A(t) = T_{a(t)}$ is a unitary operator and therefore its adjoint[18] $[A(t)]^{\dagger} = [A(t)]^{-1} = A(-t)$. Accordingly $(A(t)f, g) = (f, A(-t)g)$. Differentiating both sides of this equation with respect to t we obtain for $t = 0$,

$$(Af, g) = -(f, Ag). \tag{2.9}$$

Using this relation one then easily finds that

$$\begin{aligned} &(H_+f, g) = (f, H_- g),\ (H_3f, g) = (f, H_3g), \\ &(F_+f, g) = (f, F_- g),\ (F_3f, g) = (f, F_3g). \end{aligned} \tag{2.10}$$

A systematic use of Eq. (2.10) in (2.8) then leads to the following: If the irreducible representation $g \to T_g$ of the group L is unitary then the pair (j_0, c) characterizing it satisfies either (a) c is purely

imaginary and j_0 is an arbitrary non-negative integral or half-integral number; or (b) c is a real number in the intervals $0 \leq c \leq 1$ and $j_0 = 0$.

The representations corresponding to case (a) are called the *principal series of representations* and those corresponding to case (b) are called the *complementary series*.

3. SPINOR REPRESENTATION OF THE LORENTZ GROUP

A. The Group SL(2, C) and the Lorentz Group

In what follows we will use the fact that elements of the proper, orthochroneous, Lorentz group L can be described by means of elements of SL(2, C), the group of all 2x2 complex matrices with determinant unity. The relation between these two groups can be established as follows.

Let x_α and x'_β, with α, β = 1, 2, 3, 4, describe the coordinates of two Lorentz frames, related by

$$x'_\alpha = g_{\alpha\beta}\, x_\beta \,, \tag{3.1}$$

where $g_{\alpha\beta} \epsilon$ L. One associates with each coordinate system x_α a 2x2 Hermitian matrix Q defined by

$$Q = x_\beta \sigma^\beta, \tag{3.2}$$

where σ^k, k = 1,2,3, are the Pauli spin matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.3}$$

and σ^4 is the 2x2 unit matrix. In terms of the Q's one demands that the coordinate transformation (3.1) be expressed as

$$Q' = a\, Q\, a^\dagger, \tag{3.4}$$

where a is an element of SL(2, C), $Q' = x'_\beta \sigma^\beta$, and $a^\dagger$ is the Hermitian conjugate of a. One then finds that the relation between aϵ SL(2, C) and g ϵ L are given by[19]

$$g_{\alpha\beta} = \tfrac{1}{2}\,\mathrm{Tr}(\sigma^{\alpha} a \sigma^{\beta} a^{+}), \qquad (3.5)$$

It thus follows that the group L is homomorphic to the group SL(2,C) such that to every element $g \in L$ there correspond two matrices $\mp a \in SL(2, C)$ and, conversely, to every $a \in SL(2, C)$ there corresponds some element $g \in L$. Accordingly, the description of the representations of the group L is equivalent to that of the group SL(2,C); a representation $g \to T_g$ of L is single- or double-valued according to whether or not T_a is equal to T_{-a} or not.

B. Spinor Representation of the Group SL(2, C)

We now construct the spinor representation which contains all the irreducible finite-dimensional representations of the group SL(2, C).

We denote by P_{mn} the aggregate of all polynomials $p(z,\bar{z})$ in the variable z and its complex conjugate $\bar{z}$ of degree not exceeding m in z and n in $\bar{z}$, where m and n are fixed non-negative integers determining the representation. The space P_{mn} is a linear vector space where the operation of addition and multiplication by a number are defined in the usual way for polynomials.

An element of the group SL(2,C) will be denoted by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (3.6)$$

where a, b, c, and d are complex numbers satisfying the condition

$$ad - bc = 1.$$

Define the operator T_g in P_{mn} by

$$T_g\, p(z,\bar{z}) = (bz+d)^m(\bar{b}\bar{z}+\bar{d})^n\, p\left(\frac{az+c}{bz+d},\ \frac{\bar{a}\bar{z}+\bar{c}}{\bar{b}\bar{z}+\bar{d}}\right) \qquad (3.7)$$

The correspondence $g \rightarrow T_g$ is a linear representation of the group SL(2,C) as can be easily verified. This is the spinor representation of SL(2,C) of dimension $(m + 1)\cdot(n+1)$.

In order to relate this representation to the 2-component spinors, one realizes it in a somewhat different way.

One considers all systems of numbers $\phi_{A_1 \ldots A_m \dot{X}_1 \ldots \dot{X}_n}$, symmetrical in both the indices $A_1, \ldots A_m$ and in $\dot{X}_1, \ldots, \dot{X}_n$ taking the values 0 and 1. The set of all such systems of numbers provides a linear space, denoted by S_{mn}, of dimension $(m + 1)(n + 1)$.

A one-to-one linear mapping between the spaces P_{mn} and S_{mn} can easily be established. To each system $\phi_{A_1 \ldots A_m \dot{X}_1 \ldots \dot{X}_n} \in S_{mn}$ there corresponds the polynomial

$$p(z,\bar{z}) = \sum_{\substack{A_1,\ldots, A_m \\ \dot{X}_1,\ldots, \dot{X}_n}} \phi_{A_1 \cdots A_m \dot{X}_1 \cdots \dot{X}_n} \, z^{A_1 + \ldots + A_m} \, \bar{z}^{\dot{X}_1 + \ldots + \dot{X}_n} \qquad (3.8)$$

of degree not exceeding m in z and n in $\bar{z}$, and therefore $p(z,\bar{z}) \in P_{mn}$. On the other hand every polynomial

$$p(z,\bar{z}) = \sum_{r,s} p_{rs} \, z^r \, \bar{z}^s \qquad (3.9)$$

in P_{mn} can be written in the form (3.8) if one relate the ϕ's and p's by means of

$$\phi_{A_1 \cdots A_m \dot{X}_1 \cdots \dot{X}_n} = \frac{1}{m!\,n!} \, p_{rs} \qquad (3.10)$$

with $A_1 + \ldots + A_m = r$, and $\dot{X}_1 + \ldots + \dot{X}_n = s$.

A second form of the spinor representation is then obtained if one applies the polynomials (3.8) in Eq. (3.7). One obtains

$$T_g\, p(z,\bar{z}) = \sum_{\substack{A_1,\ldots,A_m\\ \dot{X}_1,\ldots,\dot{X}_n}} \phi'_{A_1\cdots A_m \dot{X}_1\cdots \dot{X}_n}\, z^{A_1+\cdots+A_m}\, \bar{z}^{\dot{X}_1+\cdots+\dot{X}_n} \tag{3.11}$$

where we have used the notation

$$\phi'_{A_1\cdots A_m \dot{X}_1\cdots \dot{X}_n} = \sum_{\substack{B_1\cdots B_m\\ \dot{Y}_1\cdots \dot{Y}_n}} a_{A_1B_1}\cdots a_{A_mB_m}\, \bar{a}_{\dot{X}_1\dot{Y}_1}\cdots \bar{a}_{\dot{X}_n\dot{Y}_n}\, \phi_{B_1\cdots B_m \dot{Y}_1\cdots \dot{Y}_n} \tag{3.12}$$

and where $a_{11} = a$, $a_{10} = b$, $a_{01} = c$, and $a_{00} = d$.

The quantity $\phi_{A_1\ldots A_m\dot{X}_1\ldots\dot{X}_n}$ is a spinor, symmetric in its m undotted indices and in its n dotted ones, whereas Eq. (3.12) expresses its transformation law under the matrix $a \in SL(2,C)$.

C. Infinitesimal Operators of the Spinor Representation

We now find the infinitesimal operators H_+, H_-, H_3, and F_+, F_-, F_3 of the spinor representation discussed in the last subsection.

The one-parameter subgroups of SL(2,C), corresponding to the one-parameter subgroups $a_k(t)$ and $b_k(t)$ of the group L, can easily be obtained using the formula (3.5).[20] In terms of the infinitesimal matrices a_r and b_r of the group SL(2,C) they can be written as

$$a_k(t) = \exp(t\, a_k),\quad b_k(t) = \exp(t\, b_k), \tag{3.13}$$

where $a_k = i\sigma^k/2$ and $b_k = \sigma^k/2$, and where σ^k are the Pauli spin matrices given by Eq. (3.3). Using the matrices $a_k(t)$ and $b_k(t)$ in (3.7), differentiating

both sides of the obtained equations with respect to the variable t, and putting $t = 0$, gives the expressions for the operators A_k and B_k, from which one then obtains the operators H's and F's:

$$
\begin{aligned}
H_+ &= -\frac{\partial}{\partial z} - \bar{z}^2 \frac{\partial}{\partial \bar{z}} + n\bar{z} \\
H_- &= z^2 \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} - m z \\
H_3 &= -z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} + \tfrac{1}{2}(m-n) \\
F_+ &= i\left(\frac{\partial}{\partial z} - \bar{z}^2 \frac{\partial}{\partial \bar{z}} + n\bar{z} \right) \\
F_- &= i\left(-z^2 \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} + m z \right) \\
F_3 &= i\left(z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} - \tfrac{1}{2}(m+n) \right).
\end{aligned}
\tag{3.14}
$$

4. PRINCIPAL SERIES OF REPRESENTATIONS OF SL(2,C)

A. The Hilbert Spaces $L_2^{2s}(SU_2)$ and ℓ_2^{2s}.

In its global form the principal series of representations was introduced (Naimark 1964) as a set of operators over the Hilbert space of functions $L_2^{2s}(SU_2)$, a sub-space of $L_2(SU_2)$, defined as follows.

The Hilbert space $L_2(SU_2)$ is defined as the set of all functions $\phi(u)$, where $u \epsilon SU_2$, which are measurable and satisfy the condition[21]

$$\int |\phi(u)|^2 du < \infty \tag{4.1}$$

The scalarproduct is defined by

$$(\phi_1, \phi_2) = \int \phi_1(u) \overline{\phi_2(u)} du. \tag{4.2}$$

Corresponding to any integer or half-integer s we now define a Hilbert space $L_2^{2s}(SU_2)$, which is a sub-space of $L_2(SU_2)$, as follows:

$\phi(u) \epsilon L_2^{2s}(SU_2)$ if $\phi(u) \epsilon L_2(SU_2)$ and

$$\phi(\gamma u) = e^{is\psi} \phi(u) \tag{4.3}$$

where $\gamma \epsilon SU_2$ is given by

$$\gamma = \begin{pmatrix} e^{-i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix}. \tag{4.4}$$

The scalar product is again given by Eq. (4.2).

$L_2(SU_2)$ is the direct sum of all the spaces $L_2^{2s}(SU_2)$, for all integral values of 2s.

The generalized Fourier transformation, to be introduced at the end of the section, transforms each Hilbert space $L_2^{2s}(SU_2)$ into a Hilbert space ℓ_2^{2s}, which is defined as follows (Carmeli 1970):

Consider all possible systems of numbers ϕ_m^j, where $m = -j, -j+1, \ldots, j$ and $j = |s|, |s|+1, |s|+2, \ldots$ with the condition

$$\sum_{j=|s|}^{\infty} (2j+1) \sum_{m=-j}^{j} |\phi_m^j|^2 < \infty \qquad (4.5)$$

The aggregate of all such systems ϕ_m^j forms a Hilbert space, denoted by ℓ_2^{2s}, where the scalar product is defined by

$$\sum_{j=|s|}^{\infty} (2j+1) \sum_{m=-j}^{j} \phi_m^j \overline{\psi_m^j} \qquad (4.6)$$

for any two vectors ϕ_m^j and ψ_m^j of ℓ_2^{2s}.

Let us show now that for any integral or half-integral value of s the two Hilbert spaces $L_2^{2s}(SU_2)$ and ℓ_2^{2s} are isometric, and derive the transformation between them.

Let $T_{sm}^j(u)$ be the matrix element of the irreducible representation of the group SU_2 corresponding to the eigenvalue $j(j+1)$ of the Cosimir operator J^2. The functions $T_{sm}^j(u)$ satisfy (Naimark 1964)

$$T_{sm}^j(\gamma u) = e^{is\psi} T_{sm}^j(u) \qquad (4.7)$$

and for a fixed value of s they provide a complete orthogonal set for the Hilbert space $L_2^{2s}(SU_2)$ as $m = -j, -j+1, \ldots, j$ and $j = |s|, |s|+1, |s|+2, \ldots$ (Carmeli 1969). The functions $T_{sm}^j(u)$ satisfy the orthogonality relation

$$\int T_{sm}^j(u)\, \overline{T_{s'm'}^{j'}(u)}\, du = (2j+1)^{-1} \delta_{jj'} \delta_{ss'} \delta_{mm'} \qquad (4.8)$$

Consequently, any function $\phi(u) \in L_2^{2s}(SU_2)$ can be uniquely expanded in the form

$$\phi(u) = \sum_{j=|s|}^{\infty} (2j+1) \sum_{m=-j}^{j} \phi_m^j \, T_{sm}^j(u) \qquad (4.9)$$

where

$$\phi_m^j = \int \phi(u) \overline{T_{sm}^j(u)} \, du . \qquad (4.10)$$

It can be easily shown that the system of numbers ϕ_m^j satisfies Eq. (4.5) if and only if the corresponding function $\phi(u)$ satisfies Eq. (4.1). The Hilbert spaces $L_2^{2s}(SU_2)$ and ℓ_2^{2s} are, therefore, isometric and the mapping between them is given by the "generalized Fourier transformation" (4.9) and (4.10).

B. Realization of the Principal Series of Representations in the Spaces $L_2^{2s}(SU_2)$

We are now in a position to introduce the realization of the principal series (of) representations of the group SL(2, C) in the Hilbert space $L_2^{2s}(SU_2)$. To this end we proceed as follows.[22]

Let us denote by K to aggregate of all elements k of the group SL(2,C) where k has the form

$$k = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix} \qquad (4.11)$$

with λ, μ complex numbers and $\lambda \neq 0$. One can easily verify that the set K forms a subgroup of the group SL(2, C).

We now construct the set of right cosets of the group SL(2, C) with respect to the subgroup K.

Each right coset consists of all the element kg', where g' is a fixed element of SL(2, C) and k varies over the subgroup K. Each coset will be

denoted either by Kg' or by $\widetilde{Kg'} = \widetilde{g}$ where g is an arbitrary element belonging to the coset Kg'.

It can be easily shown that every element $g \in SL(2, C)$ can be represented in the form

$$g = ku; \quad k \in K; \; u \in SU_2. \tag{4.12}$$

It follows from Eq. (4.12) that if an element $g \in SL(2,C)$ belongs to a given coset $\widetilde{g}$, then $k^{-1}g = u \in SU_2$ also belongs to the same coset. Therefore each coset $\widetilde{g}$ contains elements of the group SU_2.

Furthermore, the decomposition (4.12) is not unique since $g = ku = k'u'$;

$$k, k' \in K; \; u, u' \in SU_2 \tag{4.13}$$

where

$$k' = k\gamma \qquad\qquad u' = \gamma^{-1}u \tag{4.14}$$

with γ an arbitrary element of the subgroup Γ:

$$\Gamma: \quad \gamma = \begin{pmatrix} e^{-i\omega} & 0 \\ 0 & e^{i\omega} \end{pmatrix}, \quad \omega \text{ real}. \tag{4.15}$$

Therefore each coset $\widetilde{g}$ contains a one-parametric set of elements belonging to SU_2.

Let us denote by $u\widetilde{g}$ an arbitrary element (matrix) of the coset $\widetilde{ug} = Kug$ which belongs to SU_2. It can be proved (Naimark, 1964) that any principal series representation corresponding to the pair of parameter (s, ρ), where s is an integer or half-integer, and ρ is real, can be formulated as follows: to every element $g \in SL(2, C)$ there corresponds an operator V_g defined over the space $L_2^{2s}(SU_2)$ by

$$V_g\, \phi(u) = \frac{\alpha(ug)}{\alpha(u\bar{g})}\, \phi(u\bar{g}) \tag{4.16}$$

for all $\phi(u) \in L_2^{2s}(SU_2)$, and α is given by

$$\alpha(g) = |g_{22}|^{i\rho - 2s - 2}\, g_{22}^{2s} \tag{4.17}$$

for an arbitrary $g \in SL(2, C)$. $u\bar{g}$ is an element of the right coset $\widetilde{ug}$ defined above.

To facilitate practical applications of the representation formula (4.16) we derive now (a) an explicit expression for the matrix $u\bar{g}$ in terms of the matrices $u \in SU_2$ and $g \in SL(2,C)$. The expression will involve a phase factor which can be chosen arbitrarily; (b) the ratio $\frac{\alpha(ug)}{\alpha(u\bar{g})}$ appearing in formula (4.16) for two cases of particular interest: (i) g is unitary; (ii) g is of the form

$$g = \begin{pmatrix} \varepsilon_{22}^{-1} & 0 \\ 0 & \varepsilon_{22} \end{pmatrix}, \tag{4.18}$$

where ε_{22} is real.

(a) Let us denote the matrix $u\bar{g}$ by u'. Then u' can be written as

$$u' = \begin{pmatrix} u'_{11} & u'_{12} \\ u'_{21} & u'_{22} \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ -\bar{\beta}' & \bar{\alpha}' \end{pmatrix} \tag{4.19}$$

with the condition

$$|\alpha'|^2 + |\beta'|^2 = 1. \tag{4.20}$$

According to Eq. (4.12) ug can be written in the form $ug = k.u\ddot{g} = ku'$ where k is a matrix having the form given by Eq. (4.11). If one denotes now ug by g', then one has $g' = ku'$, or explicitly

$$\begin{pmatrix} g'_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \alpha' & \beta' \\ -\bar{\beta}' & \bar{\alpha}' \end{pmatrix}. \tag{4.21}$$

This gives

$$g'_{21} = -\lambda\overline{\beta'}\,, \qquad g'_{22} = \lambda\,\overline{\alpha'} \tag{4.22}$$

from which one obtains

$$\alpha' = \frac{\bar{g}'_{22}}{\bar{\lambda}}\,, \qquad \beta' = -\frac{\bar{g}_{21}}{\bar{\lambda}}\,. \tag{4.23}$$

Furthermore, using the condition (4.20) one obtains

$$|\lambda|^2 = |g'_{21}|^2 + |g'_{22}|^2. \tag{4.24}$$

But g' = ug. Let us denote u by

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \tag{4.25}$$

and g by

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} , \qquad (4.26)$$

then

$$\begin{pmatrix} g'_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{pmatrix} = \begin{pmatrix} \alpha g_{11} + \beta g_{21} & \alpha g_{12} + \beta g_{22} \\ -\bar{\beta} g_{11} + \bar{\alpha} g_{21} & -\bar{\beta} g_{12} + \bar{\alpha} g_{22} \end{pmatrix}. \qquad (4.27)$$

If we write now $\lambda = |\lambda| \exp(i\,\Lambda)$, where Λ is some real number (phase), then one finally obtains for (4.23) and (4.24)

$$\begin{aligned} \alpha' &= (-\beta\,\bar{g}_{12} + \alpha\,\bar{g}_{22})\,|\lambda|^{-1} e^{i\Lambda} \\ \beta' &= (\beta\,\bar{g}_{11} - \alpha\,\bar{g}_{21})\,|\lambda|^{-1} e^{i\Lambda} \end{aligned} \qquad (4.28)$$

and

$$|\lambda|^2 = |\beta\,\bar{g}_{11} - \alpha\,\bar{g}_{21}|^2 + |-\beta\,\bar{g}_{12} + \alpha\,\bar{g}_{22}|^2 . \qquad (4.29)$$

Hence, $u\bar{g}$ is determined by means of u and g up to an arbitrary phase factor.

(b) (i) let g be a unitary matrix u_o with determinant unity:

$$u_o = \begin{pmatrix} \alpha_o & \beta_o \\ -\bar{\beta}_o & \bar{\alpha}_o \end{pmatrix} ; \qquad |\alpha_o|^2 + |\beta_o|^2 = 1 \tag{4.30}$$

Then one obtains from Eqs. (4.28), (4.29)

$$\begin{aligned} \alpha' &= (-\beta\bar{\beta}_o + \alpha\alpha_o)\, e^{i\Lambda} \\ \beta' &= (\beta\bar{\alpha}_o + \alpha\beta_o)\, e^{i\Lambda} \\ |\lambda| &= 1 \end{aligned} \tag{4.31}$$

and, accordingly

$$\frac{\alpha(uu_o)}{\alpha(u\bar{u}_o)} = e^{2is\Lambda} . \tag{4.32}$$

(ii) g is the matrix given by Eq. (4.18). One then obtains

$$\begin{aligned} \alpha' &= \alpha\, \varepsilon_{22}\, |\lambda|^{-1} e^{i\Lambda} \\ \beta' &= \beta\, \varepsilon_{22}^{-1}\, |\lambda|^{-1} e^{i\Lambda} \end{aligned} \tag{4.33}$$

$$|\lambda|^2 = |\beta|^2 \varepsilon_{22}^{-2} + |\alpha|^2 \varepsilon_{22}^{2} \tag{4.34}$$

and

$$\frac{\alpha(u\varepsilon)}{\alpha(u\bar{\varepsilon})} = |\lambda|^{i\rho-2}\, e^{2is\Lambda} . \tag{4.35}$$

C. Realization of the Principal Series of Representations in the Space $\mathcal{L}_2^{2s}$.

Using the generalized Fourier transformation, introduced in Sec. 4A, we express now the representations belonging to the principal series as infinite-dimensional matrices, the elements of which will be explicitly given as integral over the group SU_2. One first notices that $T^j_{sm}(u)$ is an element of the Hilbert space $L_2^{2s}(SU_2)$. Therefore Eq. (4.16), which expresses a given principal series representation, can be applied to $T^j_{sm}(u)$ to yield

$$V_g\, T^j_{sm}(u) = \frac{\alpha(ug)}{\alpha(u\bar{g})}\, T^j_{sm}(u\bar{g}). \tag{4.36}$$

From Eqs. (4.9), (4.16), and (4.36) we have

$$V_g\, \phi(u) = \sum_j (2j+1) \sum_m \phi^j_m\, \frac{\alpha(ug)}{\alpha(u\bar{g})}\, T^j_{sm}(u\bar{g}). \tag{4.37}$$

Since $\dfrac{\alpha(ug)}{\alpha(u\bar{g})}\; T^j_{sm}(u\bar{g})$ is a vector in the Hilbert space $L_2^{2s}(SU_2)$ it can be expanded as a series of the form (4.9). One obtains

$$\frac{\alpha(ug)}{\alpha(u\bar{g})}\, T^j_{sm}(u\bar{g}) = \sum_{j'} (2j'+1) \sum_{m'} V^{jj'}_{mm'}(g;s,\rho)\, T^{j'}_{m'}(u) \tag{4.38}$$

where, because of Eq. (4.10)

$$V^{jj'}_{mm'}(g;s,\rho) = \int \frac{\alpha(ug)}{\alpha(u\bar{g})}\, T^j_{sm}(u\bar{g})\, \overline{T^{j'}_{sm'}(u)}\, du \tag{4.39}$$

Combining Eqs. (4.37), (4.38) one finally obtains

$$V_g \phi(u) = \sum_j (2j+1) \sum_m \phi'^j_m \; T^j_{sm}(u) \tag{4.40}$$

where

$$\phi'^{j'}_{m'} = \sum_{j=|s|}^{\infty} (2j+1) \sum_{m=-j}^{j} V^{jj'}_{mm'}(g; s, \rho)\, \phi^j_m . \tag{4.41}$$

Thus the operator V_g of the principal series of representations of SL(2, C) in the space $\mathcal{L}_2^{2s}$ is the linear transformation determined by Eq. (4.41) describing the law of transformation of the quantities ϕ^j_m where $j = |s|, \; |s| + 1, \; |s| + 2, \ldots$ and $m = -j, -j + 1, \ldots, j$. The coefficients $V^{jj'}_{mm'}$ $(g; s, \rho)$ are functions of $g \in SL(2, C)$ and ρ and s, where ρ is real and 2s is an integer. These functions are the matrix elements of an infinite-dimensional matrix, whose rows are labeled by (j, m) and columns - by (j', m'). They are given by Eq. (4.39) as integrals over the group SU_2.

It will be noted that the quantities ϕ^j_m, whose transformation law is given by Eq. (4.41), were obtained from the representation formula (4.16), in analogy with the way 2-component spinors, transforming according to Eq. (3.12), both being coefficients appearing in the spaces of representations.

D. Comparison with the Infinitesimal Approach

We have seen in the present section that all the irreducible representations of the group SL(2, C) belonging to the principal series are characterized by a pair of numbers (s, ρ) where s is an integer or half-integer and ρ is real. If the representation is given in a global form, the space of the representation depends on the value of s (see Sec. 4A) and the operators depend on both s and ρ (Eqs. (4.16), (4.17)).

The principal series was already defined in terms of the infinitesimal operators in Sec. 2C. It was found to depend on a pair of papameters (j_0, c) where j_0 takes the values 0, $\frac{1}{2}$ 1, 3/2,... and c is pure imaginary. The values of these parameters occurred in the formulae for the infinitesimal operators (Eq. (2.8)).

By applying the global form of a given representation to infinitesimal elements of the group SL(2, C) one can calculate the infinitesimal operators of the representation. By comparing the infinitesimal operators thus obtained with the results of Sec. 2 one establishes the relationship between the pairs of parameters (s, iρ) and (j_0, c). The result is as follows:

For $j_0 = 0$, one obtains

$$s = 0 , \qquad c = \pm i \frac{\rho}{2}, \tag{4.42}$$

and for $j_0 \neq 0$, one obtains

$$\begin{aligned} j_0 &= s & c &= -i\frac{\rho}{2} \text{ if } s > 0 \\ j_0 &= s & c &= i \frac{\rho}{2} \text{ if } s < 0 . \end{aligned} \tag{4.43}$$

5. COMPLEMENTARY SERIES OF REPRESENTATIONS OF SL(2 C)

A. Realization of the Complementary Series of Representations in the Space H.

In Sec. 4 the principal series of representations, which is unitary and irreducible, was realized as sets of operators on the Hilbert spaces L_2^{2s} (SU_2). The scalar product was given simply by Eq. (4.2) and the operators were defined by Eqs. (4.16), (4.17).

The principal series of representations, however, do not realize all irreducible unitary representations of the group SL(2, C). Rather, every irreducible unitary representation of the group SL(2, C) is unitarily equivalent to a representation of either the principal series or the complementary series of representations.

Formally, the complementary series of representations formulae can be obtained from that of the principal series formuale (4.16) if one takes $\rho = i\sigma$ and $s = 0$ in the latter and assume that now σ is real and has the values $0<\sigma<2$ (Naimark 1954). Unfortunately, the operators thus defined are not unitary in the scalar product (4.2): Eq. (4.16) defines a unitary operator if and only if $\alpha(g)$ is defined by Eq. (4.17) with ρ real.

A realization of the complementary series representations in terms of unitary operators is, however, possible on the Hilbert space of H_σ to be defined as follows.

Let H denote the set of all bounded measurable functions $\phi(u)$, where u is an element of SU_2, satisfying the condition

$$\phi(\gamma u) = \phi(u), \tag{5.1}$$

and where $\gamma \in SU_2$ is given by Eq. (4.4). [The condition (5.1) is in fact identical with (4.3) for the case $s = 0$.] Introduce in H the scalar product

$$<\phi_1, \phi_2> = \pi \iint K(u' u''^{-1}) \, \phi_1(u') \, \phi_2(u'') \, du' \, du'' \tag{5.2}$$

for $\phi_1, \phi_2 \in H$. Here $K(u' u''^{-1})$ is a kernel function defined by

$$K(u) = |u_{21}|^{\sigma-2}, \tag{5.3}$$

where $0 < \sigma < 2$ and the integral on the right hand side of Eq. (5.2) converges absolutely. The space H can be shown to be Euclidean, whose completion (which is a Hilbert space[23]) we denote by H_σ.

In the Hilbert space H_σ, the operators V_g of a representation of the complementary series, defined in complete analogy with the principal series, are unitary. Explicitly, the definition of V_g is as follows:

$$V_g \, \phi(u) = \frac{\alpha(ug)}{\alpha(u\bar{g})} \, \phi(u\bar{g}) \tag{5.4}$$

where $\phi \in H$ and $\alpha(g)$ is given by

$$\alpha(g) = |g_{22}|^{\sigma-2} \tag{5.5}$$

for any $g \in SL(2, C)$ and $0 < \sigma < 2$. The representations thus defined are irreducible and unitary.

B. Orthogonal Set in the Space H

We now define a set of functions which provides an orthogonal basis in the space H. It is given by

$$t^j_m(u) = N_j \, T^j_{om}(u) \tag{5.6}$$

for $j = 0, 1, 2, 3\ldots$ and $m = -j, -j+1, \ldots, j$, where the $T^j_{om}(u)$ were defined in Sec. 4C and N_j is a real normalization factor whose value is given by

$$N_j = \left\{ \pi \int K(u) \, T^j_{oo}(u) \, du \right\}^{-\frac{1}{2}}. \tag{5.7}$$

The integration involved in the definition of N_j can be carried out to yield an expression of the normalization constants as a finite sum of Euler B - functions. The result is (Carmeli and Malin 1971)

$$N_j^{-2} = \pi \sum_{m=o}^{j} (-)^{3j-m} \binom{j}{m}^2 B\left(m+1, j+\frac{\sigma}{2}-m\right) \tag{5.8}$$

where[24]

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}. \tag{5.9}$$

To show that t^j_m indeed provide an orthogonal basis in H we calculate the scalar product

$$< t^{j_1}_{m_1}, t^{j_2}_{m_2} > = \pi \iint K(u'u''^{-1})\, t^{j_1}_{m_1}(u')\, \overline{t^{j_2}_{m_2}(u'')}\, du'\, du'' \tag{5.10}$$

$$= \pi N_{j_1} N_{j_2} \iint K(u'u''^{-1})\, T^{j_1}_{om_1}(u')\, du'\, \overline{T^{j_2}_{om_2}(u'')}\, du''$$

By making the transition u' → u' u" in the integral (5.10) one obtains

$$< t^{j_1}_{m_1}, t^{j_2}_{m_2} >$$

$$= \pi N_{j_1} N_{j_2} \iint K(u')\, T^{j_1}_{om_1}(u'u'')\, du'\, \overline{T^{j_2}_{om_2}(u'')}\, du'' \tag{5.11}$$

Using the relation

$$T^{j_1}_{om_1}(u'u'') = \sum_{m=-j_1}^{j_1} T^{j_1}_{om}(u')\, T^{j_1}_{mm_1}(u'') \tag{5.12}$$

in the last integral we obtain

$$< t^{j_1}_{m_1}, t^{j_2}_{m_2} >$$

$$= \pi N_{j_1} N_{j_2} \sum_{m=-j}^{j} \int K(u')\, T^{j_1}_{om}(u')\, du' \int T^{j_1}_{mm_1}(u'')\, \overline{T^{j_2}_{om_2}(u'')}\, du'' \tag{5.13}$$

Using now the orthogonality relation (4.8) that the matrices T^j satisfy, we obtain

$$< t^{j_1}_{m_1}, \ t^{j_2}_{m_2} >$$

$$= \pi N_{j_1} N_{j_2} \left\{ \int K(u') \, T^{j_1}_{00}(u') du' \right\} \frac{\delta^{j_1 j_2} \delta^{m_1 m_2}}{2j_1 + 1} \qquad (5.14)$$

which, by virtue of Eq. (5.7) gives

$$< t^{j_1}_{m_1} , t^{j_2}_{m_2} > = \frac{\delta^{j_1 j_2} \ \delta^{m_1 m_2}}{2j_1 + 1} . \qquad (5.15)$$

C. Realization of the Complementary Series in the Space h

In analogy with the generalized Fourier transformation, introduced in Sec. 4A, between the space $L^{2s}_2(SU_2)$ and ℓ^{2s}_2, there exists for the complementary series a transformation from the Euclideon space of functions H (and its completion, the Hilbert space H_σ) to a Euclideon space of systems of numbers h (and its completion, the Hilbert space h_σ) (Carmeli and Malin 1971).

The Euclidean space h is defined as the aggregate of all systems of numbers ψ^j_m, where m = -j, -j + 1, ..., j and j = 0, 1, 2, ..., satisfying

$$\sum_j (2j+1) \ N_j^{-2} \sum_{m=-j}^{j} |\psi^j_m|^2 < \infty \qquad (5.16)$$

The scalar product is defined by

$$\sum_j (2j+1)\, \mathcal{N}_j^{-2} \sum_{m=-j}^{j} \phi_m^j\, \overline{\psi_m^j} \qquad (5.17)$$

for any two vectors ϕ_m^j and ψ_m^j in h. The coefficients N_j are defined by Eq. (5.7).

In analogy with Eqs. (4.9) and (4.10) relating the space $L_2^{2s}(SU_2)$ and ℓ_2^{2s} the correspondence between H and h is given by

$$\phi(u) = \sum_j (2j+1)\, \mathcal{N}_j^{-1} \sum_m \phi_m^j\, t_m^j(u) \qquad (5.18)$$

and

$$\phi_m^j = \mathcal{N}_j \langle \phi, t_m^j \rangle, \qquad (5.19)$$

where t_m^j was defined by Eq. (5.6). Comparing

$$\langle \phi, \psi \rangle = \sum_j (2j+1)\, \mathcal{N}_j^{-2} \sum_m \phi_m^j\, \overline{\psi_m^j} \qquad (5.20)$$

with Eq. (5.18) we see that $\phi(u) \in H$ if and only if the corresponding $\phi_m^j \in h$.

If we denote now by h_σ the completion[23] of the Euclidean space h, then the isometric mapping (5.18), (5.19) of H on h can be extended in a unique way by continuity to an isometric mapping of H_σ on h_σ. The operators V_g of a representation of the complementary series in the space H_σ pass over into operators in the space h_σ, which are also denoted by V_g and whose explicit expression we find below.

Applying Eq. (5.4) to the t_m^j gives

$$V_g\, t_m^j(u) = \frac{\alpha(ug)}{\alpha(u\bar{g})}\, t_m^j(u\bar{g}) \tag{5.21}$$

Using this result in Eq. (5.18) yields

$$V_g\, \phi(u) = \sum_j (2j+1) N_j^{-1} \sum_m \phi_m^j \frac{\alpha(ug)}{\alpha(u\bar{g})}\, t_m^j(u\bar{g}) \tag{5.22}$$

Expanding $\frac{\alpha(ug)}{\alpha(u\bar{g})}\, t_m^j\,(u\bar{g})$in the series (5.18) we obtain

$$\begin{aligned} V_g\, \phi(u) &= \\ &= \sum_j (2j+1) \sum_m \phi_m^j \sum_{j'} (2j'+1) N_{j'}^{-1} \sum_{m'} V_{mm'}^{jj'}(g;\sigma)\, t_{m'}^{j'}(u) \end{aligned} \tag{5.23}$$

where, because of Eqs. (5.19) and (5.2)

$$V_{mm'}^{jj'}(g,\sigma) = \pi \frac{N_{j'}}{N_j} \iint K(u'u''^{-1}) \frac{\alpha(u'g)}{\alpha(u'\bar{g})}\, t_m^j(u'\bar{g})\, \overline{t_{m'}^{j'}(u'')}\, du'\, du'' \tag{5.24}$$

Accordingly, Eq. (5.23) has the form

$$V_g \psi(u) = \sum_j (2j+1) \mathcal{N}_j^{-1} \sum_m \phi'^{j}_{m} t^j_m(u) \qquad (5.25)$$

where

$$\phi'^{j'}_{m'} = \sum_{j=0}^{\infty} (2j+1) \sum_{m=-j}^{j} V^{jj'}_{mm'}(g;\sigma)\, \phi^j_m \, . \qquad (5.26)$$

Eq. (5.26) defines a linear transformation in the space h_σ corresponding to the operator V_g of the complementary series. $V^{jj'}_{mm'}(g, \sigma)$, which are given by Eq. (5.24) as double integrals over the group SU_2, are functions of $g \in SL(2,C)$ and σ where $0 < \sigma < 2$. These functions are the matrix elements of an infinite-dimensional matrix, where rows are labeled by (j, m) and columns - by (j', m').

D. Comparison with the Infinitesimal Approach

The complementary series in its global form, as defined in this section, is characterized by a parameter σ, whose range of variation is $0 < \sigma < 2$. The value of σ determines the scalar product (Eqs. (5.2) and (5.3)) in the Hilbert space of representations and also the operators of the representations (eqs. (5.4) and (5.5)).

The complementary series was defined in Sec. 2 through the infinitesimal approach. All the irreducible representations of the group SL(2, C) were characterized in Sec. 2 by a pair of numbers (j_0, c), where j_0 takes the values 0, $\frac{1}{2}$, 1 3/2,... and c is complex. The complementary series representations were characterized by $j_0 = 0$, $0 < c < 1$.

To establish the relationship between the parameters σ and c one applies the global form of a given representation to infinitesimal elements of the group SL(2,C) and compares the infinitesimal elements thus obtained with the results of Sec. 2. The result is

$$j_0 = 0, \quad c = \pm (\sigma/2). \tag{5.27}$$

6. COMPLETE SERIES OF REPRESENTATIONS OF SL(2, C)

A. Realizations of the Complete Series in the Spaces $L_2^{2s}(SU_2)$ and ℓ_2^{2s}.

As has already been pointed out in Sec. 5 all the unitary representations of the group SL(2, C) are included in either the principal or the complementary series.[25] Gel'faund and Naimark (1947) and Naimark (1954, 1964) have shown that all the completely irreducible[26] representations of SL(2, C) (i.e. not necessarily unitary) are included, up to equivalence, in a series of representations known as the <u>complete series</u>.[27]

We define here the complete series and its realization in the spaces $L_2^{2s}(SU_2)$ and ℓ_2^{2s}.

All the representations of the complete series can be characterized by a pair of numbers (s, ρ) where s is an integer or half-integer and ρ satisfies $\rho^2 \neq -4(|s| + k)^2$, $k = 1, 2, 3, \ldots$ and is otherwise an arbitrary complex number. The pairs (s, ρ) and $(-s, -\rho)$ define the same representation.

All the representations of the complete series can be realized in the spaces $L_2^{2s}(SU_2)$, defined in Sec. 4A. The space of realization depends therefore, on s alone and is independent of ρ. A given representation corresponding to a pair (s, ρ) is realized in $L_2^{2s}(SU_2)$ by a set of operators V_g, $g \in SL(2, C)$ defined by

$$V_g \phi(u) = \frac{\alpha(ug)}{\alpha(u\bar{g})} \phi(u\bar{g}) \qquad (6.1)$$

for $\phi(u) \in L_2^{2s}(SU_2)$, where

$$\alpha(g) = g_{22}^{2s} |g_{22}|^{i\rho - 2s - 2} \qquad (6.2)$$

and $u\bar{g}$ was defined in Sec. 4B.

These formulas are the same as Eq. (4.16) and (4.17) for the principal series; the difference is that now ρ can take complex values, while in Eqs. (4.18) ρ is real. It can be shown that the operators V_g defined by Eqs. (6.1), (6.2) are unitary if and only if ρ is real.

In complete analogy with Sec. 4C the generalized Fourier transformation, introduced in Sec. 4A, can now be utilized to obtain a realization of the complete series in the spaces $\mathcal{L}_2^{2s}$. The result is

$$\phi'^{j'}_{m'} = \sum_{j=|s|}^{\infty} (2j+1) \sum_{m=-j}^{j} V^{jj'}_{mm'}(g;s,\rho)\, \phi^{j}_{m} \tag{6.3}$$

where

$$V^{jj'}_{mm'}(g;s,\rho) = \int \frac{\alpha(ug)}{\alpha(u\bar{g})}\, T^{-j}_{sm}(u\bar{g})\, \overline{T^{j'}_{m'}(u)}\, du \tag{6.4}$$

which is again the same as Eqs. (4.39), (4.41) except insofar as the definition of α (g)(Eq. (6.2)) is extended to include complex values of ρ.

B. Relation to the Principal and Complementary Series

The complete series describes all the infinite-dimensional completely irreducible representations, to within equivalence, of the group SL(2, C). The meaning of equivalence here is such that the spaces of two equivalent representations need not be isometric, but it is the formulas which are essential for the representations and not the norm of the space. In the present subsection we define equivalence of representations and show that the representations belonging to the complementary series are, from this point of view, equivalent to representations contained in the complete series.

The definition of equivalence between representations realized in Banach spaces requires some preliminary mathematical definitions:

(i) the group ring X. Let X deonte the set of all infinitely differentiable functions x(g), $g \in$ SL(2, C), which vanish for all the matrices g satisfying

$$|g_{11}|^2 + |g_{12}|^2 + |g_{21}|^2 + |g_{22}|^2 > C \qquad (6.5)$$

for a big enough number C which may depend on the function x(g). This set forms a ring if addition and multiplication by complex numbers are defined in the usual way and multiplication of ring elements is defined as follows:[20]

$$x_1 \cdot x_2 (g) = \int x_1(g') x_2 (g'^{-1} g) \, dg' \qquad (6.6)$$

(ii) Conjugate representations. Given a Banach space B, whose elements are deonted by ξ , its conjugate space B is defined as the space of all bounded linear functionals f (ξ) in B.

Given an operator T in B its conjugate operator T' is defined in B' as

$$T'f(\xi) = f(T\xi) \tag{6.7}$$

Now, given a representation in terms of operators V_g on a Banach space B we define

$$\tilde{V}_g = V'_{g-1} \tag{6.8}$$

as the conjugate representation in the Banach space B'.

(iii) The set Ω corresponding to a given representation in a Banach space B is defined as the aggregate of all finite linear combinations of the vector $V_x\xi$ (V_x is defined in footnote 28) where $\xi \in M^j$ (see Sec. 2) for any value of j, and $x \in X$. The set corresponding to the conjugate representation is denoted by Ω'.

Following Naimark (1964) we now define two representations V_g^1, V_g^2 on Banach spaces B^1, B^2 as equivalent if there exists linear operators A^1 and A^2 from B^1 to B^2 and from B^2 to B^1 respectively, whose domains of definition are Ω^1, Ω^2 and domains of variation Ω^2, Ω'^2 respectively, satisfying, for all $\xi \in \Omega$, $f \in \Omega'^2$,

$$f(A^1\xi) = A^2 f(\xi); \tag{6.9}$$

if $A^1\xi = 0$, $A^2 f = 0$ then $\xi = 0$, $f = 0$,

$$A' V_x \xi = V_x^2 A\xi \quad ; \tag{6.10}$$

$$A^2 V_x^2 f = V_x^1 A^2 f \, . \tag{6.11}$$

It is noteworthy that for the representations to be equivalent the Banach spaces need not be isometric.

In the previous subsection the complete series representations were characterized by a pair of paremeters (s, ρ) where s in an integer or half-integer, and ρ is a complex number. We will now show that the complete series representations characterized by $s = 0$ and ρ satisfying $0 < -i\rho < 2$ are equivalent to the complementary series representations.

The space of representations of the complete series representations corresponding to $s = 0$ was defined in the previous subsection as the Hilbert space $L_2^o\,(SU_2)$. The space of representations of the complementary series was defined as the Hilbert space H_σ. These spaces correspond to B^1, B^2 respectively in the definition of equivalence. Now, the crucial point is this: if V_g^1 is a complementary series representation corresponding to a value σ of the parameter, and V_g^2 is a complete series representation corresponding to the values $s = 0$, $\rho = i\sigma$, then the representation V_g^1, V_g^2 are given in the two Banach spaces by the same formula (Eq. (6.1), (6.2) and Eqs. (5.4), (5.5) respectively). It follows now that the sets Ω^1, Ω^2 corresponding to a given representation in the Banach spaces $L_2^o(SU_2)$ and H_σ are the same, because both consists of all the finite linear combinations of the vectors $V_x\xi$ where ξ is any of the functions $T^j_{om}(u)$. (V_x was defined in footnote 25).

The operators A^1, A^2 in the definition of equivalence are trivially defined now as the identity operators in $\Omega^1 = \Omega^2$ and $\Omega'^1 = \Omega'^2$ respectively. One can easily check that they satisfy Eqs. (6.9) - (6.11). Therefore any complementary series representation, corresponding to a value σ is equivalent to the complete series representation characterized by the pair of parameters $s = 0$, $\rho = i\sigma$.

C. Relation to Spinors

In introducing the complete series we restricted the values of its parameter (s, ρ) by excluding the representations for which

$$\rho^2 = -4 (|s| + k)^2; \quad k = 1, 2, 3, \ldots \tag{6.12}$$

We now consider the representations corresponding to Eq. (6.12) and show that:

(i) the representations realized by the general formula for the complete series, Eqs. (6.1), (6.2), are not irreducible if Eq. (6.12) is satisfied;

(ii) when the general formulas (6.1), (6.2) of the complete series apply to a finite-dimensional linear space of polynomials over SU_2, instead of an infinite-dimensional Hilbert space, they realize the spinor representations;

(iii) the generalized Fourier transform of these polynomials is related to the standard form of 2-component spinors by a linear transformation, which is explicitly derived.

(i) To see that indeed when $\rho^2 = -4 (|s| + k)^2$ the representation (6.1) is not irreducible we proceed as follows.

Suppose that $\rho = -2i(|s| + k)$ and denote by P_{MN} the set of all homogeneous polynomials in u_{21}, $\bar{u}_{21}$, u_{22} and $\bar{u}_{22}$:

$$p(u) = \sum_{\alpha, \beta, \gamma, \delta} u_{21}^{\alpha} \, \bar{u}_{21}^{\beta} \, u_{22}^{\gamma} \, \bar{u}_{22}^{\delta} \tag{6.13}$$

with the conditions

$$\alpha - \beta + \gamma - \delta = 2s \tag{6.14}$$

$$\alpha + \beta + \gamma + \delta = 2|s| + 2k - 2 \tag{6.15}$$

where k = 1, 2, 3, ... One can easily see, using Eq. (6.14) that

$$p(\gamma u) = e^{is\psi} p(u) \tag{6.16}$$

where γ is given by (4.4). Therefore P_{MN} is a subspace of the Hilbert space L_2^{2s} (SU_2). We show that P_{MN} is invariant with respect to the operator V_g of Eq. (6.1). To this end one writes

$$g = u_1 \epsilon\, u_2 \ , \tag{6.17}$$

where u_1, $u_2 \in SU_2$ and ϵ is given by

$$\epsilon = \begin{pmatrix} \epsilon_{22}^{-1} & 0 \\ 0 & \epsilon_{22} \end{pmatrix} \tag{6.18}$$

with ϵ_{22} a real number. Since $V_g = V_{u_1} V_\epsilon V_{u_2}$, it is sufficient to show

that P_{MN} is invariant under each of the operators V_{u_1}, V_ϵ and V_{u_2}. Now

$$V_{u_1} p(u) = \frac{\alpha(uu_1)}{\alpha(u\bar{u}_1)} p(u\bar{u}_1) \tag{6.19}$$

It is shown in Sec. 4B that $\alpha(uu_1) / \alpha(\bar{u}\bar{u}_1)$ is equal to $\exp(2 is \Lambda)$, where Λ is an arbitrary real number. Also, a direct calculation, using Eq. (4.31) shows that

$$\begin{aligned} p(uu_1) = \sum_{\alpha,\beta,\gamma,\delta} e^{i\Lambda(-\alpha+\beta-\gamma+\delta)} & a_{\alpha\beta\gamma\delta} \\ & \times (uu_1)_{21}^{\alpha} \, (\overline{uu_1})_{21}^{\beta} \, (uu_1)_{22}^{\gamma} \, (\overline{uu_1})_{22}^{\delta} \end{aligned} \tag{6.20}$$

Hence, using the condition (6.14) one obtains

$$V_{u_1} p(u) = p(uu_1) \tag{6.21}$$

which shows that P_{MN} is invariant with respect to the operator V_{u_1} (and, of course, to V_{u_2}).

Similarily, P_{MN} is invariant with respect to V_ϵ, where

$$V_\epsilon p(u) = \frac{\alpha(u\epsilon)}{\alpha(u\bar{\epsilon})} p(u\bar{\epsilon}). \tag{6.22}$$

In Sec. 4B it is shown that $\alpha(u\epsilon) / \alpha(u\bar{\epsilon})$ is equal to $\exp. (2 is \Lambda) |\lambda|^{i\rho - 2}$, where $|\lambda|$ is given by Eq. (4.34). Furthermore, one easily verifies that

$$p(u\varepsilon) = \sum_{\alpha,\beta,\gamma,\delta} e^{i\Lambda(-\alpha+\beta-\gamma+\delta)} |\lambda|^{-(\alpha+\beta+\gamma+\delta)} \varepsilon_{22}^{-\alpha-\beta+\gamma+\delta}$$

$$\times\ a_{\alpha\beta\gamma\delta}\, u_{21}^{\alpha}\, \bar{u}_{21}^{\beta}\, u_{22}^{\gamma}\, \bar{u}_{22}^{\delta}\ . \tag{6.23}$$

Using the conditions (6.14) and (6.15) and the fact that $\rho = -2i(\,|s| + k)$, one finds

$$V_{\varepsilon}\, p(u) = \sum_{\alpha,\beta,\gamma,\delta} \varepsilon_{22}^{-\alpha-\beta+\gamma+\delta}\, a_{\alpha\beta\gamma\delta}\, u_{21}^{\alpha}\, \bar{u}_{21}^{\beta}\, u_{22}^{\gamma}\, \bar{u}_{22}^{\delta}\ . \tag{6.24}$$

This shows that $V_{\epsilon}p\ (u)$ is a polynomial in the space P_{MN}. Hence Γ_{MN} is invariant with respect to the operator V_g, and therefore the representation (6.1) is not irreducible when $\rho = -2i(\,|s\,| + k)$, $k = 1, 2, 3, \ldots.$[29]

(ii) We now show that the operators defined by Eqs. (6.1), (6.2) which realize all the infinite-dimensional irreducible representations of SL(2, C), realize the spinor representations as well, if the space of the representations is properly defined as a space of polynomials over SU_2.

Starting from Eq. (3.9) let us denote $p(z, \bar{z})$ by $f(z)$ and let

$$\alpha\,(g) = g_{22}^{m}\ \bar{g}_{22}^{n} \tag{6.25}$$

where

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{6.26}$$

is an element of SL(2, C). Equation (3.7) can then be written in the form

$$T_g f(z) = \alpha(zg) f(z\bar{g}). \tag{6.27}$$

Here z denotes a complex variable and also the matrix

$$z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \tag{6.28}$$

and the matrix $z' = z\bar{g}$ amounts to a transformation in which the variable z goes over into the new variable

$$z' = g'_{21} / g'_{22} , \tag{6.29}$$

where the matrix $g' \in SL(2, C)$ is given by

$$g' = zg = \begin{pmatrix} g_{11} & g_{12} \\ g_{11} z + g_{21} & g_{12} z + g_{22} \end{pmatrix}. \tag{6.30}$$

So that the new variable z' , according to (6.29) and (6.30) is given by

$$z' = \frac{g_{11} z + g_{21}}{g_{12} z + g_{22}} . \tag{6.31}$$

If now we write

$$\phi(u) = \pi^{1/2} \alpha(u) f(z) \tag{6.32}$$

where u, z, $\in \tilde{Z}$,[30] and $z = u_{21}/u_{22}$, then

$$\phi(u) = \pi^{1/2} \sum_{r=0}^{m} \sum_{s=0}^{n} p_{rs}\, u_{21}^{r}\, u_{22}^{m-r}\, \bar{u}_{21}^{s}\, \bar{u}_{22}^{n-s} . \tag{6.33}$$

Hence ϕ (u) runs through all polynomials which are homogeneous in u_{21}, u_{22} of degree m and in $\bar{u}_{21}$, $\bar{u}_{22}$ of degree n, and p_{rs} are related to spinors by (3.10). Let $\tilde{P}_{mn}$ denote the set of all such polynomials. Then $\tilde{P}_{mn}$ is the set of all polynomials homogeneous of degree m + n in u_{21}, u_{22}, $\bar{u}_{21}$, $\bar{u}_{22}$, satisfying the condition

$$\phi(\gamma u) = e^{i(m-n)\psi/2}\, \phi(u) , \tag{6.34}$$

where γ is given by Eq. (6.14). The operators of the representation in the space $\tilde{P}_{mn}$ are then given by the formula

$$T_g\, \phi(u) = \frac{\alpha(ug)}{\alpha(u\bar{g})}\, \phi(u\bar{g}) , \tag{6.35}$$

where ϕ (u) $\in \tilde{P}_{mn}$ and $u\bar{g}$ is a matrix of SU_2 whose explicit expression is given in Sec. 4B. Comparison of (6.25) with (6.2) gives

$$m = \frac{i}{2}\rho + n - 1 , \qquad n = \frac{i}{2}\rho - n - 1 . \tag{6.36}$$

We have, in fact, obtained already this space of polynomial in part (i) of the present section as that subspace of $L_2^{2s}(SU_2)$ which is invariant under the representation. Indeed, using Eqs. (6.14); (6.15) and (6.36) one obtains

$$\gamma = m - \alpha , \qquad \delta = n - \beta . \tag{6.37}$$

Eq. (6.13) can now be written as

$$p(u) = \sum_{\alpha=0}^{m} \sum_{\beta=0}^{n} a_{\alpha\beta} \, u_{21}^{\alpha} \, \bar{u}_{21}^{\beta} \, u_{21}^{m-\alpha} \, \bar{u}_{22}^{n-\beta} . \tag{6.38}$$

Comparing (6.38) with (6.33) we see that $a_{\alpha\beta}$ is just $\pi^{\frac{1}{2}} p_{rs}$. Hence $a_{\alpha\beta}$ is related to spinors, by (3.10), by

$$a_{\alpha\beta} = \pi^{1/2} m! \, n! \, \phi_{A_1 \cdots A_m \dot{X}_1 \cdots \dot{X}_n} \tag{6.39}$$

with

$$A_1 + A_2 + \ldots + A_m = \alpha , \qquad \dot{X}_1 + \dot{X}_2 + \ldots + \dot{X}_n = \beta .$$

and the representations (6.35) is indeed a realization of the spinor representations.

(iii) We are now in a position to find the connection between spinors and the generalized Fourier transform ϕ^j_m in the finite-dimensional case. Since $p(u) \in L_2^{2s}(SU_2)$, one can expand it into a finite series in $T^j_m(u)$

$$p(u) = \sum_{j=|s|}^{J} (2j+1) \sum_{m=-j}^{j} \phi^j_m \, T^j_{sm}(u) \qquad (6.40)$$

where ϕ^j_m is related to $p(u)$ by

$$\phi^j_m = \int p(u) \, \overline{T^j_{sm}(u)} \, du. \qquad (6.41)$$

Using the expression (6.38) for $p(u)$ in (6.41) one obtains

$$\phi^j_m = \sum_{\alpha=0}^{M} \sum_{\beta=0}^{N} \tilde{C}^{jMN}_{m\alpha\beta} \, a_{\alpha\beta} \qquad (6.42)$$

where $\tilde{C}^{jMN}_{m\alpha\beta}$ are some numerical coefficients,

$$\tilde{C}^{jMN}_{m\alpha\beta} = \int \overline{T^j_{sm}(u)} \, u_{21}^{\alpha} \, \bar{u}_{21}^{\beta} \, u_{22}^{M-\alpha} \, \bar{u}_{22}^{N-\beta} \, du. \qquad (6.43)$$

And in terms of 2-component spinors, by Eq. (6.39), one obtains

$$\phi_m^j = \sum_{\alpha=0}^{M} \sum_{\beta=0}^{N} C_{m\alpha\beta}^{jMN} \phi_{A_1 \cdots A_M \dot{X}_1 \cdots \dot{X}_N} \tag{6.44}$$

where

$$C_{m\alpha\beta}^{jMN} = \pi^{1/2} M! N! \tilde{C}_{m\alpha\beta}^{jMN} \tag{6.45}$$

Here $A_1 + \ldots + A_M = \alpha$, $\dot{X}_1 + \ldots + \dot{X}_N = \beta$.

The generalized Fourier transform ϕ_m^j is, therefore, related to the spinors $\phi_{A_1 \ldots A_M \dot{X}_1 \ldots \dot{X}_N}$ via a linear transformation, given explicitly by Eqs. (6.43), (6.45) as an integral over the group SU_2.

ACKNOWLEDGEMENTS AND CONCLUDING REMARKS

It is a pleasure to thank Professor A. O. Barut for suggestions and critical comments. We also wish to thank Mr. J. Berning for his help in organizing the material. Finally, we would like to apologize to the authors, whose contributions were not referred to. The literature on the Lorentz group and its applications is so extensive that some selectivity, based on the authors' background and interests is unavoidable.

FOOTNOTES

1. See, for example, the session on infinite-dimensional representations of particles in Hagen, Guralnik, and Mathur (1967).
2. See, for example, the monograph of Streater and Wightman (1964).
3. See, Nambu (1966, 1967a); Barut and Kleinert (1967a, b); Barut, Corrigan, and Kleinert (1968a, b); Fronsdal (1967a, b); Bohm (1967); Takabayashi (1967); Abarbanel and Frishman (1968); Chodos (1970); Chodos and Haymaker (1970); Humi and Malin (1969); Noga (1970); Kursunoglu (1968); Aghassi, Roman, and Sanbilli (1970); Komar and Slad (1969); Bisiacchi, Colucci, and Fronsdal (1969).
4. One of the first systems which was described by infinite-dimensional wave equations was the non-relativistic H-atom (Fronsdal 1967b; Barut and Kleinert 1967c, d, e; Nambu 1967b; Kleinert 1968). More recently, an equation which describes the relativistic H-atom was obtained by Barut and Baiquni (1969a, b). The relation of the Majorana equation to the two-dimensional Schrodinger equation was also investigated by Biedenharn and Giovannini (1967), Morita (1969), Barut and Duru (1971).
5. See, Domokos, Kovesi-Domokos, and Mansouri (1970a, b); Domokos, Kovesi-Domokos, and Schonberg (1970); Bacry and Nuyts (1967); Watanabe and Miyazaki (1969); Matsumoto (1970); Morita (1970); Bars and Gursey (1971).

6. See also Barut's review of hadron symmetries (Barut 1970). The relationship between current algebra and infinite dimensional equations were recently reviewed by O'Raifeartaigh (1969) and Niederer and O'Raifeartaigh (1970).

7. Spinors have also been of great importance in general relativity theory. For reviews of applications of spinors in general relativity see Penrose (1960), Pirani (1965), and Carmeli and Fickler (1972).

8. For details see Streater and Wightman (1964).

9. These matrices are given by

$$a_1(\Psi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Psi & -\sin\Psi & 0 \\ 0 & \sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ldots,$$

and

$$b_1(\Psi) = \begin{pmatrix} \cosh\psi & 0 & 0 & \sinh\psi \\ 0 & & & 0 \\ 0 & & & 0 \\ \sinh\psi & 0 & 0 & \cosh\psi \end{pmatrix}, \ldots.$$

10. The a_r and b_r are related to $a_r(\Psi)$ and $b_r(\Psi)$ by

$$a_r(\Psi) = \exp(\Psi a_r),\ b_r(\Psi) = \exp(\Psi b_r),$$

and are given by

$$a_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ldots, b_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ldots .$$

11. $A_r(\Psi)$ and $B_r(\Psi)$ are continuous functions of Ψ and are called basic one-parameter groups of operators for the given representation. They satisfy the relations $A_r(\Psi_1)\ A_r(\Psi_2) = A_r(\Psi_1 + \Psi_2)$, $B_r(\Psi)\ B_r(\Psi_2) = B_r(\Psi_1 + \Psi_2)$. $A_r(0) = 1$, $B_r(0) = 1$. If the representation is finite-dimensional then the operators $A_r(\Psi)$ and $B_r(\Psi)$ are differentiable functions of Ψ. If the representation is infinite-dimensional, however, these operators might be non-differentiable (see footnote 13).

12. $A_r(\Psi)$ and $B_r(\Psi)$ might then be expanded in terms of A_r and B_r as $A_r(\Psi) = \exp(\Psi A_r)$, $B_r(\Psi) = \exp(\Psi B_r)$.

13. In general, let $A(t)$ be a continuous one-parameter group of operators in a Banach space R, and denote by $X(A)$ the set of all vectors $x \in R$ for which the limit of $(A(t)\,x-x)/t$, when $t \to 0$ exists in the sense of

tho norm in R. Obviously the set $X(A)$ contains the vector $x = 0$. Define now the operator A for all $x \in X(A)$ by $Ax = \lim\{(A(t)\ x - x)/t\}$ at the limit $t \to 0$. The domain of definition, $X(A)$, of the operator A is a subspace of R, and A is linear, i.e., $A(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1\ A\ x_1 + \lambda_2\ A\ x_2$ for x_1, $x_2 \in X(A)$. Such an operator A is called the infinitesimal operator of the one-parameter group $A(t)$. If $A(t) = T_{a(t)}$ is the group of operators of the representation $g \to T_g$, corresponding to a one-parameter subgroup $a(t)$ of the group L, the corresponding operator A is then called the infinitesimal operator of the representation $g \to T_g$.

14. The superscript in f_m^j specifies the subspace uniquely since each irreducible representation of O_3 is contained at most once in any given irreducible representation of the group L.

15. Eqs. (2.8), for unitary representations case and under certain assumptions, were first obtained by Gel'fand (see Naimark (1964), p. 117); they later were rederived by Harish-Chandra (1947a, 1947b), and by Gel'fand and Iaglom (1948).

16. For the physical significance of non-unitary representations see Barut and Malin (1968).

17. A representation $g \to T_g$ of a group G in a space R is called unitary if R is a Hilbert space and T_g is a unitary operator for all $g \in G$. This implies that $(T_g x, T_g y) = (x,y)$ for all $g \in G$ and all $x, y \in R$, where (x,y) denotes the scalar product in R.

18. An operator B is called an adjoint to the operator A if $(Ax, y) = (x, By)$ for all $x, y \in R$.

19. Compare the analoguous formulas for the rotation group given by Eqs. (2.10) and (2.11) in Carmeli (1968). Eqs. (3.5) can easily be proved by finding the value of the expression $(\frac{1}{2})\ \mathrm{Tr}\ (\sigma^\alpha a \sigma^\beta a^t) x_\beta = (\frac{1}{2})\ \mathrm{Tr}(\sigma^\alpha a \sigma^\beta x_\beta a^t) = (\frac{1}{2})\ \mathrm{Tr}\ (\sigma^\alpha a Q a^t) = (\frac{1}{2})\ \mathrm{Tr}\ (\sigma^\alpha Q') = (\frac{1}{2})\ \mathrm{Tr}\ (\sigma^\alpha \sigma^\beta x'_\beta) = (\frac{1}{2})\ \mathrm{Tr}\ (\sigma^\alpha \sigma^\beta) x'_\beta = \delta^{\alpha\beta} x'_\beta = x'_\alpha = g_{\alpha\beta} x_\beta$.

20. These matrices for the group SL(2, C) are given by

$$a_1(t) = \begin{pmatrix} \cos\frac{t}{2} & i\sin\frac{t}{2} \\ i\sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}, \qquad b_1(t) = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}$$

$$a_2(t) = \begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}, \qquad b_2(t) = \begin{pmatrix} \cosh\frac{t}{2} & i\sinh\frac{t}{2} \\ -i\sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix},$$

$$a_3(t) = \begin{pmatrix} e^{\frac{it}{2}} & 0 \\ 0 & e^{-\frac{it}{2}} \end{pmatrix}, \qquad b_3(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

21. The integral in Eq. (4.1) and throughout this paper are invariant integrals over the group SU_2 which satisfy the conditions

$$\int f\,(uu_1)\,du = \int f\,(u_1u)\,du = \int f\,(u)\,du$$

for any $u_1 \in SU_2$, and

$$\int f\,(u^{-1})\,du = \int f\,(u)\,du$$

$$\int du = 1.$$

22. For a different form of realization of the principal series see e.g. Gel'fand, Grae. and Vilenkin (1966).

23. Every Euctidean space can be completed to a Hilbert space. See e.g. Naimark (1959, 1964); Lyusternik and Sobolov (1951).

24. These functions were recently used by Veneziano (1968) for the construction of crossing-symmetric, Regge-behaved scattering amplitude for linearly rising trajectories.

25. It is interesting to note that the definition of the principal and complementary series of representations can be generalized from the group SL(2, C) to SL(N, C) for arbitrary $N > 2$. However, for $N > 2$ there exist, in general unitary representations not contained in either the principal or the complete series (Stein, 1967).

26. The definition of complete irreducibility is as follows (Naimark 1964): given a representation V_g of the group SL(2, C) on a Banach space B one first defines a bounded linear operator C as admissible if it has the form

$$C\,\xi = \sum_{i=1}^{n} f_i\,(\xi)\, e_i$$

where $f_1, \ldots, f_n \in \Omega'$ and $e_1, \ldots, e_n \in \Omega$. The definitions of the sets Ω, Ω' are given in Sec. 6B. One then defines the representations as completely irreducible if for every admissible operator C in B there exists a sequence $x_n \in X$ such that $(V_{x_n} \xi, \eta) \to (C\,\xi\,\eta)$ as $n \to \infty$ for all $\xi \in \Omega_1$ $\eta \in \Omega'$. X is the group ring, defined in Sec. 6B, and the operators V_{x_n} are defined in footnote 28. It can be shown that every unitary irreducible representation in a separable Hilbert space is completely irreducible.

27. For a definition of equivalence of representations in the sense of the present section see Sec. 6B.

28. This definition comes about as follows: given a representation of SL(2,C) as a set of operators V_g one defines an operator V_x corresponding to every function $x(g) \in X$ as follows:

$$V_x = \int x(g)\; V_g\; d\,g$$

By straightforward calculation one finds that

$$V_{x_1}\; V_{x_2} = V_{x_1 x_2}$$

if $X_1 x_2$ is defined by Eq. (6.7). For further details see Naimark (1964).

29. The representation (6.1) is not irreducible also when $\rho = 2i(|s| + k)$, where $k = 1, 2, 3, \ldots$, since the pairs (s, ρ) and $(-s, -\rho)$ define the same completely irreducible representation.

30. $\tilde{z}$ is the set of all matrices kg, where g is an element of $SL(2, C)$, fixed, and k varies through the entire group of matrices of the form given by Eq.(4.11). For more details see Naimark (1964), p.140.

REFERENCES

Abarbanel, H. and Frishman, Y., 1968, Phys. Rev. 171, 1442.

Abors, E., Grodsky, I. T. and Norton, R. E., 1967, Phys. Rev. 159, 1222.

Aghassi, J. J., Roman P. and Santilli, R. M., 1970, Phys. Rev. 1D, 2753.

Bacry, H. and Nuyts, J., 1967, Phys. Rev. 157, 1471.

Bargmann, V., 1947, Ann. Math. 48, 568.

Bars, I. and Gursey, F., 1971, Phys. Rev. D4, 1769.

Barut, A. O., 1970, Rapporteur's talk given at the XV-th International Conference on High Energy Physics, Kiev; University of Colorado preprint.

Barut, A. O., and Baiquni, A., 1969a, Phys. Rev. 184, 1342.

———, 1969b, Phys. Lett. 30A, 352.

Barut, A. O., Corrigan, D. and Kleinert, H., 1968a, Phys. Rev. Lett. 20, 167.

———, 1968b, Phys. Rev. 167, 1527.

Barut, A. O., and Duru, I. H., 1971, to be published.

Barut, A. O., and Kleinert, H. 1967a, Phys. Rev. Lett. 18, 754.

———, 1967b, Phys. Rev. 156, 1546.

———, 1967c, Phys. Rev. 156, 1541.

———, 1967d, Phys. Rev. 156, 1180.

———, 1967e, Phys. Rev. 160, 1149.

Barut, A. O., and Kom n, G. J., 1970, Phys. Rev. D1, 418.

Barut, A. O., and Malin, S., 1968a, Rev. Mod. Phys. 40, 632.

———, 1968b, Nuovo Cim. 58A, Series X, 835.

———, 1971, Ann. Phys. to be published.

Bebie, H., and Lentwyler, H., 1967, Phys. Rev. Lett. 19, 618.

Biedenhan, L. C., and Giovannini, A., 1967, Nuovo Com. 51A, 870.

Bisiacchi, J., Calucci, J., and Fronsdal, C., 1969, Acta Phys. Hungar. 26, 21.

Bohm, A., 1967, Phys. Rev. 158, 1408.

Bohm, A., 1968, Lectures in Theoretical Physics, Vol. XB, Gordon and Breach, New York.

Carmeli, M., 1968, J. Math Phys. 9, 1987.

———, 1969, J. Math. Phys. 10, 569.

———, 1970, J. Math. Phys. 11, 1917.

Carmeli, M., and Fickler, S. I., 1972, Phys. Rev. D 5, 290.

Carmeli, M., and Malin, S., 1971 , J. Math Phys. 12, 225.

———, 197 , to be published.

Chang, S. J., Dashen, R., and O'Raifeartaigh, L., 1969a, Phys. Rev. 182, 1805.

———, 1969b, Phys. Rev. 182, 1819.

Chodos, A., 1970, Phys. Rev. D1, 2937.

Chodos, A., and Haymaker, R. W., 1970, Phys. Rev. 2D, 793.

Cocho, G., Fronsdal, C., and White, R., 1969, Phys. Rev. 180, 1547.

Dashen, R., and Gell-Mann, M., 1966, Phys. Rev. Lett. 17, 340.

Domokos, G., Kovesi-Domokos, S., and Mansouri, F., 1970a, Phys. Rev. D1, 500.

———, 1970b, Phys. Rev. D1, 512.

Domokos, G., Kovesi-Domokos, S., and Schonberg E., 1970, Phys. Rev. D2, 1026.

Feldmann, G., and Matthews, P. T., 1967a, Phys. Rev. 154, 1241.

———, 1967b, Phys. Rev. 155, 1176.

Fronsdal, C., 1967a, Phys. Rev. 156, 1654.

———, 1967b, Phys. Rev. 156, 1665.

———, 1967c, Phys. Rev. 156, 1653.

Fronsdal, C., and Harun Ar-Rashid, A. M., 1969, Trieste preprint.

Gel'fand, I. M., Graev, I., and Vilenkin, N. Ya., 1966, Generalized Functions Vol. 5: Integral Geometry and Representations Theory, Academic Press, New York and London.

Gel'fand, I. M. and Iaglom, A. M., 1948, Zh. Eksp. Theor. Fiz. 18, 703. English translation: technical translation TT-345, National Research Council of Canada, Ottawa, 1953.

Gel'fand, I. M., and Naimark, M. A., 1946, J. Phys. X, 93.

———, 1947, Izv. Akad. Nauk. SSSR, Ser. Matem. 11, 411.

Gell-Mann, M., Horn, D., and Weyers, J., 1967, Proceedings of the Heidelberg International Conference on Elementary Particles.

Grodsky, I. T., and Streater, R. F., 1968, Phys. Rev. Lett. 20, 695.

Hagen, C. R., Guralnik, G., and Mathur, V. S., 1967, editors, Proceedings of the International Conference on Particles and Fields, Rochester 1967, (Wiley-Interscience, New York).

Hamprecht, B., and Kleinert, H., 1969, Phys. Rev. 180, 1410.

Harish-Chandra, 1947a, Proc. Roy. Soc. A 189, 372.

———, 1947b, Phys. Rev. 71, 793.

———, 1951, Proc. Nat. Acad. Sci. 37, 813.

———, 1954, Trans. Am. Math. Soc. 76, 485.

Humi, M., and Malin, S., 1969, Phys. Rev. 187, 2778.

Katz, J., and Noga, M., 1970, Phys. Rev. D2, 1657.

Kleinert, H., 1968, Lectures in Theoretical Physics, Vo. XB, Gordon and Breach.

Kleinert, H., Corrigan, D., and Hemprecht, B., 1970, Springer Tracts in Mod. Phys. 49.

Komar, A. A., and Slad, L. M., 1969, J. Math. Phys. USSR 1, 50.

Kursunoglu, B., 1968, Phys. Rev. 167, 1452.

Lam, Y., 1968, Ann. Phys. 50, 323.

Lentwyler, H., 1968, Phys. Rev. Lett. 20, 561.

Lyusternik, L. A., and Sobalov, V. I., 1951, Elements of Functional Analysis (Moscow).

Majorana, E., 1932, Nuovo Cim. 9, 335.

Matsumoto, Y., 1970, Prog. Theor. Phys. 43, 499.

Miyazaki, T., 1968a, Prog. Theor. Phys. 38, 285.

———, 1968b, Prog. Theor. Phys. 40, 427.

———, 1969, Prog. Theor. Phys. 42, 125.

———, 1969, Prog. Theor. Phys. 42, 346.

———, 1970a, Prog. Theor. Phys. 43, 1616.

———, 1970b, University of Tokyo preprint UT-64.

———, 1970c, Prog. Theor. Phys. 44, No. 3.

Morita, K., 1969, Nagoya Univ. Preprint .

———, 1970, Prog. Theor. Phys. 43, 847.

Nambu, Y., 1966, Prog. Theor. Phys. Suppl. 37 and 38, 368.

———, 1967a, Phys. Rev. 160, 1171.

———, 1967b, Proc. Int. Conference on Particles and Fields, Interscience.

Naimark, M. A., 1954, Dokl. Akad. Nauk SSSR 97, 969.

———, 1959, Normed Rings, P. Noordhaff Ltd. Groninger, the Netherlands.

———, 1964, Linear Representations of the Lorentz Group, Pergamnon Press, Inc., New York.

Niederer, U. H., and O'Raifeartaigh, L., 1970, Dublin preprint.

Noga, M., 1970, Phys. Rev. 2D, 304.

Oksak, A. I., and Todorov, I. T., 1969, Comm. Math. Phys. 14, 271.

———, 1970, Phys. Rev. D1, 3511.

O'Raifeartaigh, 1969, Proc. First Coral Gables Conference on Fundamental Interactions, Gordon and Breach.

Penrose, R., 1960, Ann. Phys.(N.Y.) 10, 171.

Pirani, F. A. E., 1965, Lectures on General Relativity, Prentice-Hall Inc., Englewood Cliffs, New Jersey.

Stein, E. M., 1967, Ann. Math. 86, 461.

Streater, R. F., and Wightman, A. S., 1964, PCT, Spin and Statistics and All That, Benjamin, New York.

Takabayashi, T., 1967, Prog. Theor. Phys. 38, 966.

Talman, J. D., 1968, Special Functions: A Group Theoretic Approach, based on lectures by E. P. Wigner, W. A. Benjamin, New York.

Veneziano, G., 1968, Nuovo Cim. 57A, 190.

Watanabe, T., and Miyazaki, T., 1969, Prog. Theor. Phys. 42, 121.

LIST OF SYMBOLS

Symbol	Description
ε_{ijk}	Levi-Civita skew-symmetric tensor with $\varepsilon_{123} = 1$
F_+, F_-, F_3	Infinitesimal generators of the Lorentz group
g	Element of the Lorentz group L or the group SL(2, C)
H_+, H_-, H_3	Infinitesimal generators of the Lorentz group
H, h	Euclidean spaces
H_σ, h_σ	Hilbert spaces
L	Proper orthochroneous Lorentz group
$L_2^{2s}(SU_2)$	Hilbert space
$L_2(SU_2)$	Hilbert space
ℓ_2^{2s}	Hilbert space
N_j	Real normalization factor
O_3	Three-dimensional pure rotation group
SL(2, C)	Group of all 2x2 complex matrices with determinant unity
SU_2	Group of all 2x2 unitary matrices with determinant unity
$T^j_{mn}(u)$	Matrix elements of irreducible representations of the group SU_2
$t^j_m(u)$	Orthogonal set of functions
T_g	Operator
u	Element of the group SU_2
V_g	Operator
x_1, x_2, x_3, x_4	Space-time coordinates